27/10/22 MATH 4030 Tutorial

Reminders:

- Midterms graded,

  Avg: ~58

- Assignment 4 - due 3/11 11:59pm.
  - Do not need to complete Question 1.

Principal Curvatures / Types of Points:

- $K(p) = \det S_p = k_1 k_2$ diagonalizing $S_p = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$.
- $H(p) = \frac{1}{2} \operatorname{tr} S_p = \frac{1}{2}(k_1 + k_2)$.

These eigenvalues are called principal curvatures

e_1, e_2. eigenvectors are called principal directions

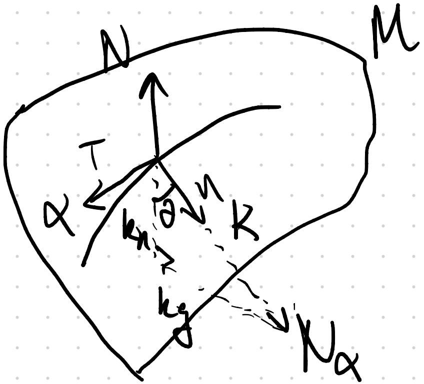
Normal curvature: N - unit normal vector field of M. $\alpha(s)$. $\alpha' = T$, $n(s) = N(s) \times T(s)$

positively oriented $\{T, n, N\}$

Then we can write T' in the basis $\{n, N\}$

$$T' = \underset{\text{geodesic curvature}}{k_g} n + \underset{\text{normal curvature}}{k_n} N$$

$k_n = k\langle N_\alpha, N\rangle = k\cos\theta$ where θ is the angle between N_α, N.



$\alpha'(0) = v$. Then $k_n(0) = I\!I_p(v,v)$

e_1, e_2 principal directions are an onbs of T_pM

$v = e_1\cos\varphi + e_2\sin\varphi$

$$\begin{aligned} k_n = I\!I_p(v,v) &= \langle S_p(v), v\rangle \\ &= \langle S_p(e_1\cos\varphi + e_2\sin\varphi), e_1\cos\varphi + e_2\sin\varphi\rangle \\ &= \langle S_p(e_1\cos\varphi) + S_p(e_2\sin\varphi), e_1\cos\varphi + e_2\sin\varphi\rangle \\ &= \langle k_1e_1\cos\varphi + k_2e_2\sin\varphi, e_1\cos\varphi + e_2\sin\varphi\rangle \\ &= k_1\cos^2\varphi + k_2\sin^2\varphi. \end{aligned}$$

φ - angle from e_1 to v.

← Euler's Formula.

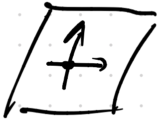
all normal curvatures k_n satisfy

$k_1 \le k_n \le k_2$.

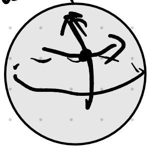
Types of Points: $p \in M$

- Elliptic if $K(p) = \det S_p > 0$ (principal curvatures $\neq 0$, same sign).
  e.g. points on spheres are elliptic.
- Hyperbolic if $K(p) = \det S_p < 0$ (principal curvatures $\neq 0$, different sign).
- Parabolic if $K(p) = \det S_p = 0$ but $S_p \neq 0$. (only one of $k_1, k_2 = 0$).
- Planar if $S_p = 0$ ($k_1 = k_2 = 0$).
- Umbilical if $k_1 = k_2$ (including $k_1 = k_2 = 0$).

Planar



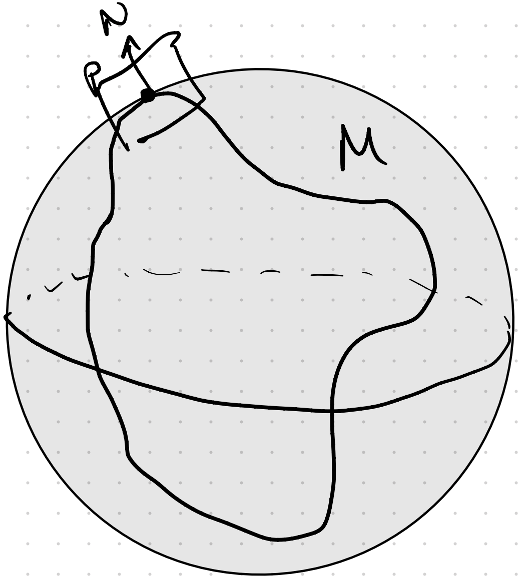
Umbilical



do Carmo 3-3 Q16:

Show that a surface which is compact has an elliptic point.

Pf: let M be a compact surface in $\mathbb{R}^3$ (closed, bounded). By boundedness, $M \subset B_R(0)$. I can decrease R continuously until it is tangent to M at some point, say p.



Clearly at p, $K_B(p) = \frac{1}{\|p\|^2} > 0$.

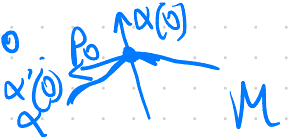
$M \subset B_R$, and we can show that $K_M \geq K_B$.

So $K_M(p) \geq K_B(p) = \frac{1}{\|p\|^2} > 0$. So p is elliptic.

alternatively, $f: M \to \mathbb{R}$ by $f(p) = \|p\|^2$. Then since M is compact $f(M)$ is compact and hence has a maximum p_0.

let α be a smooth curve $\alpha(0) = p_0$, then since p_0 is a maximum, we have

- $\frac{d}{dt} f(\alpha(t))\Big|_{t=0} = 0 \Rightarrow 0 = 2\alpha'(0)\cdot\alpha(0) \Rightarrow$ the vector in $\mathbb{R}^3$ $\alpha(0)$ is normal to M at p_0



- $\frac{d^2}{dt^2} f(\alpha(t))\Big|_{t=0} \le 0 \Rightarrow \alpha''(0)\cdot\alpha(0) + \alpha'(0)\cdot\alpha'(0) \le 0$.

Let N be the unit normal vector of M at p_0 given by $N = \frac{\alpha(0)}{|\alpha(0)|}$, then we have

$$\alpha''(0)\cdot\alpha(0) + \alpha'(0)\cdot\alpha'(0) = \alpha''(0)\cdot N|\alpha(0)| + \alpha'(0)\cdot\alpha'(0)$$

$$= |\alpha(0)|\langle -dN(\alpha'(0)), \alpha'(0)\rangle + \alpha'(0)\cdot\alpha'(0)$$

$$= |\alpha(0)|\langle S_p(\alpha'(0)), \alpha'(0)\rangle + \langle\alpha'(0), \alpha'(0)\rangle \le 0.$$

Since only condition on α was that $\alpha(0) = p_0$, we can take $\alpha = \gamma_i$ where $\gamma_i(0) = 0$ and $\gamma_i'(0)$ are the principal directions (i.e. eigenvectors of S_p) for $i = 1, 2$. Then we have $\langle\gamma_i'(0), \gamma_i(0)\rangle = 1$ and above inequality becomes

$$k_i|\gamma_i(0)| + 1 \le 0 \Rightarrow k_i \le \frac{-1}{|\gamma(0)|} < 0$$ for both $i = 1, 2$. Hence $k_1 k_2 > 0$ and we are done.

Cor: Show that there exist no compact minimal surfaces ($H \equiv 0$) in $\mathbb{R}^3$.

Pf: Suppose M compact and $H \equiv 0$

$H \equiv 0 \Rightarrow \frac{1}{2}(k_1 + k_2) \equiv 0 \Rightarrow k_1, k_2$ have opposite signs everywhere or zero everywhere.

↑ plane

$\Rightarrow$ M has no elliptic pts. → contradiction!

do Carmo 3-2 Q1:

Show that at a hyperbolic point, the principal directions bisect the asymptotic directions.

Recall: v is an asymptotic direction at p if $k_n = 0$

Hint: Use Euler's formula.

let v_1 be an asymptotic direction. Then by Euler's formula: $0 = k_1 \cos^2\theta + k_2 \sin^2\theta$

$\Rightarrow -k_1 \cos^2\theta = k_2 \sin^2\theta \Leftrightarrow \tan^2\theta = \frac{-k_1}{k_2} \geq 0$ since k_1, k_2 have different signs (hyperbolic pt.).

So θ has two solutions $\pm \arctan\sqrt{\frac{-k_1}{k_2}} = \pm\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

So there exist exactly 2 asymptotic directions v_1, v_2. and $\angle(v_1, v_2) = 2\alpha$.

and e_1, e_2 are the axes corresponding to $\theta = 0$, $\theta = \pi$, so we see that they bisect v_1, v_2.